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Static Planar Domain Wall in General Relativity with a Cosmological Constant

B. Linet[1]

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By finding all static metrics with plane symmetry describing static planar walls, we prove the existence of a static planar domain wall in the case of a negative cosmological constant. This metric and its source can be justified as a limit case within the framework of a simple scalar field, the thin-wall approximation being obtained by a numerical analysis.

1. INTRODUCTION

Within the context of the spontaneously broken gauge symmetries, the cosmological effects of domain walls have been a subject of considerable investigation. Therefore, some recent works have focused on solutions to Einstein equations without cosmological constant describing the gravitational field of a planar wall, i.e., a surface layer with plane symmetry. A vacuum domain wall in the thin-wall approximation is a planar wall in which the tension is equal to the surface energy density. These time-dependent solutions have been considered by Vilenkin (1983) and Ipser and Sikivie (1984). More generally, all solutions for planar walls which are reflection symmetric with respect to the wall have been determined (Ipser, 1984).

We confine our present work to the static planar walls in general relativity with a cosmological constant Λ. We find all static metrics which are reflection symmetric with respect to the wall. In the case of a negative cosmological constant, there exists a static planar domain wall which has its surface energy density σ related to the cosmological constant by the expression

$$4\pi\sigma = 2\left(-\frac{\Lambda}{3}\right)^{1/2} \tag{1}$$

[1]Unité Associée au C.N.R.S. No. 769, Université P. et M. Curie, Institut Henri Poincaré, 11, rue P. et M. Curie, 75231 Paris Cedex 05, France.

where we have chosen units in which $G = c = 1$. The space-time outside the domain wall is locally anti-de Sitter.

As was pointed out by Zeldovich et al. (1975) in the Minkowskian space-time, a static planar domain wall can be obtained as a limit case within the framework of a simple model with a scalar field acquiring vacuum expectation values $\pm\eta$. They consider the static solution φ with plane symmetry describing the transition between $\varphi = -\eta$ and $\varphi = \eta$ to the field equation deduced from the potential

$$V(\varphi) = \lambda^2(\varphi^2 - \eta^2)^2 \tag{2}$$

A more general approach is discussed by Kibble (1976).

In curved space-time, our method is modeled after the work of Zeldovich et al. (1975) by considering the static solutions with plane symmetry to Einstein equations having such a scalar field as source. However, the existence of the solution describing the transition between $\varphi = -\eta$ and $\varphi = \eta$ will be based only on a numerical analysis as well as the thin-wall approximation in which $1/\lambda\eta$ goes to zero.

2. STATIC PLANAR WALLS

The static solutions with plane symmetry to Einstein equations with a cosmological constant have been analysed (Kramer et al., 1980). It is possible to show that, in the coordinate system in which the metric has the general form

$$ds^2 = f^2(x)\, dt^2 - \frac{1}{f^2(x)}\, dx^2 - g^2(x)(dy^2 + dz^2) \tag{3}$$

the general solution has the following expression:

$$\begin{aligned} f^2(x) &= \frac{C}{x - x_0} - \frac{\Lambda}{3}(x - x_0)^2 \\ g^2(x) &= K^2(x - x_0)^2 \end{aligned} \tag{4}$$

where C and K are two constants such that there exists a range of the coordinate x in which

$$\frac{C}{x - x_0} - \frac{\Lambda}{3}(x - x_0)^2 > 0 \tag{5}$$

We remark with the following:

(i) If $C \neq 0$, by changing the coordinate systems in which metric (3) keeps the same form, C and K can take arbitrary values. Hence we have only a space-time locally described by metric (3).

(ii) If $C = 0$, which is only possible for $\Lambda < 0$ by virtue of inequality (5), it is easy to see that the space-time is locally anti-de Sitter. Hereafter we assume that $x_0 > 0$ and $K = 1$.

In this coordinate system, we consider a static planar wall located at $x = 0$. Requiring that the space-time is reflection symmetric with respect to the wall, the metrics describing the gravitational field of the wall are

$$ds^2 = f^2(|x|)\, dt^2 - \frac{1}{f^2(|x|)}\, dx^2 - g^2(|x|)(dy^2 + dz^2) \tag{6}$$

which are well defined in a neighborhood of $x = 0$ if inequality (5) is satisfied at $x = 0$, i.e.,

$$-\frac{\Lambda}{3} - \frac{C}{x_0^3} > 0 \tag{7}$$

The nonvanishing components of the energy-momentum tensor of the static planar wall are

$$T_t^t = \sigma \frac{1}{(g_{xx})^{1/2}} \delta(x) \qquad \text{and} \qquad T_y^y = T_z^z = \tau \frac{1}{(g_{xx})^{1/2}} \delta(x) \tag{8}$$

where σ is the surface energy density and τ the tension. We suppose that the energy-momentum tensor (8) satisfies the properties that every observer measures a positive energy density and a timelike energy-momentum density. For an algebraic form (8), we must assume that

$$\sigma > 0 \qquad \text{and} \qquad -\sigma \le \tau \le \sigma \tag{9}$$

Making use of Einstein equations, the discontinuity at $x = 0$ of the first derivative of the functions $f^2(|x|)$ are $g^2(|x|)$ will give the quantities σ and τ. We find

$$\begin{aligned} 8\pi\sigma &= 4\left(-\frac{\Lambda}{3} - \frac{C}{x_0^3}\right)^{1/2} \\ 8\pi\tau &= \frac{4\Lambda/3 + C/x_0^3}{\Lambda/3 + C/x_0^3}\left(-\frac{\Lambda}{3} - \frac{C}{x_0^3}\right)^{1/2} \end{aligned} \tag{10}$$

Hence we have obtained a set of metrics (6) characterized by a parameter C/x_0^3, restricted by inequality (7), describing static planar walls with energy-momentum tensor (8).

It is most convenient to characterize these space-times by the parameter τ/σ restricted by the following inequalities dependent on the sign of the cosmological constant:

$$\begin{aligned} -1 \le \tau/\sigma < 1/4 \qquad & \text{for } \Lambda > 0 \\ 1/4 < \tau/\sigma \le 1 \qquad & \text{for } \Lambda < 0 \end{aligned} \tag{11}$$

the value $\tau/\sigma = 1/4$ is only possible if $\Lambda = 0$ and we obtain the set of metrics found by Horsky (1968). We point out that the important case $\tau = \sigma$ corresponding to a domain wall occurs for $\Lambda < 0$ with $C = 0$; we then have expression (1) for the energy surface density and the space-time is locally anti-de Sitter outside the wall.

3. THE STATIC PLANAR DOMAIN WALL

We now consider as Zeldovich et al. (1975) a simple scalar field having an action

$$S = \int (-g)^{1/2}[\tfrac{1}{2}g^{\alpha\beta}\,\partial_\alpha\varphi\,\partial_\beta\varphi - \lambda^2(\varphi^2 - \eta^2)^2]\,d^4x \tag{12}$$

its energy-momentum tensor being

$$T^\alpha_\beta = g^{\alpha\gamma}\,\partial_\gamma\varphi\,\partial_\beta\varphi - \delta^\alpha_\beta[\tfrac{1}{2}g^{\mu\nu}\,\partial_\mu\varphi\,\partial_\nu\varphi - \lambda^2(\varphi^2 - \eta^2)^2] \tag{13}$$

Firstly, we look for the static solutions with plane symmetry to Einstein equations with source (13) which have the reflection symmetry with respect to the hypersurface $x = 0$ and so that the scalar field satisfies

$$\varphi(x) = -\varphi(-x) \tag{14}$$

We have proved that these solutions may be written in a coordinate system in which

$$ds^2 = e^{2u(x)}\,dt^2 - dx^2 - e^{2u(x)}(dy^2 + dz^2) \tag{15}$$

where the function u and the scalar field φ, are determined uniquely in a neighborhood of $x = 0$ by the following system of differential equations:

$$\begin{gathered}\frac{d^2u}{dx^2} = -4\pi\left(\frac{d\varphi}{dx}\right)^2\\ \left(\frac{d\varphi}{dx}\right)^2 = 2\lambda^2(\varphi^2 - \eta^2)^2 + \frac{3}{4\pi}\left(\frac{du}{dx}\right)^2 + \frac{\Lambda}{4\pi}\end{gathered} \tag{16}$$

assuming the initial conditions $u(0) = (du/dx)(0) = \varphi(0) = 0$. We suppose

$$\Lambda > -8\pi\lambda^2\eta^4 \tag{17}$$

Secondly, we require furthermore that φ describes a transition between $\varphi = -\eta$ and $\varphi = \eta$. In general a solution to system (9) does not verify the condition

$$\varphi \to \eta \qquad \text{as } x \to \infty \tag{18}$$

except for a particular value of the cosmological constant. To show this, we introduce the notation

$$
\begin{aligned}
&X = \lambda \eta x \\
&\varphi(x) = \eta Y(x) \qquad \text{and} \qquad \frac{du}{dx}(x) = \lambda \eta Z(x)
\end{aligned}
\tag{19}
$$

The system (16) becomes

$$
\begin{aligned}
&\frac{dZ}{dX} = -4\pi\eta^2 \left(\frac{dY}{dX}\right)^2 \\
&\left(\frac{dY}{dX}\right)^2 = 2(Y^2-1)^2 + \frac{3}{4\pi\eta^2} Z^2 + \frac{E}{4\pi\eta^2}
\end{aligned}
\tag{20}
$$

where we have put $E = \Lambda/\lambda^2\eta^2$; the initial conditions are $Y(0) = Z(0) = 0$. We study the solution to systems (20) in function of the parameter $4\pi\eta^2$. We expect that (i) for a value of $4\pi\eta^2$ given, there exists a value $E(4\pi\eta^2)$ of E, always negative, so that

$$
Y \to 1 \qquad \text{and} \qquad Z \to -\left[-\frac{E(4\pi\eta^2)}{3}\right]^{1/2} \qquad \text{as } X \to \infty \tag{21}
$$

(ii) for this solution, the integral

$$
I(4\pi\eta^2) = \int_{-\infty}^{+\infty} \left[\frac{1}{2}\left(\frac{dY}{dX}\right)^2 + (Y^2-1)^2\right] dX \tag{22}
$$

is finite and we have the following limits:

$$
\begin{aligned}
&I(4\pi\eta^2) \to 4\sqrt{2}/3 \\
&I(4\pi\eta^2) \to \frac{2}{4\pi\eta^2}\left[-\frac{E(4\pi\eta^2)}{3}\right]^{1/2} \qquad \text{as } 4\pi\eta^2 \to 0
\end{aligned}
\tag{23}
$$

These expectations are based on our numerical analysis of the system of differential equations (20). We have not a formal proof of results (21) and (23).

To summarize, we note the existence of a static transition with plane symmetry between $\varphi = -\eta$ are $\varphi = \eta$ if the value of the cosmological constant is

$$
\Lambda = \lambda^2 \eta^2 E(4\pi\eta^2) \tag{24}
$$

Moreover the exterior space-time is asymptotically anti-de Sitter. On the other hand, the quantity

$$
\sigma = \int_{-\infty}^{+\infty} T_t^t \, dx = \int_{-\infty}^{+\infty} \left[\frac{1}{2}\left(\frac{d\varphi}{dx}\right)^2 + \lambda^2(\varphi^2 - \eta^2)^2\right] dx \tag{25}
$$

can be expressed in the form

$$\sigma = \lambda\eta^3 I(4\pi\eta^2) \tag{26}$$

Taking into account results (23), we obtain that

$$\sigma \to \lambda\eta^3 4\sqrt{2}/3 \qquad \text{as } 4\pi\eta^2 \to 0 \tag{27}$$

Curiously, in the limit where $4\pi\eta^2$ tends to zero, the expression of σ in terms of parameters λ and η coincides with the one found by Zeldovich et al. (1975) in the Minkowskian space-time. We also obtain that

$$4\pi\sigma \to 2\left(-\frac{\Lambda}{3}\right)^{1/2} \qquad \text{as } 4\pi\eta^2 \to 0 \tag{28}$$

In order to σ be finite, we must simultaneously suppose that $1/\lambda\eta$ goes to zero but this is just the thin-wall approximation. In this limit, metric (15) corresponding to a transition between $\varphi = -\eta$ and $\varphi = \eta$ tends to the metric describing the static planar domain wall.

4. CONCLUSION

We have found all static planar walls which are characterized by the parameter τ/σ whose the range of variation depends on the sign of the cosmological constant according to inequalities (11). A static planar domain wall $\tau/\sigma = 1$ exists in the case of a negative cosmological constant but it seems to have academic interest more especially as the energy surface density is directly related to the cosmological constant by formula (1). However, we have an illustrative example in general relativity in which we have justified the domain wall within the framework of a simple scalar field in the thin-wall approximation as in the Minkowskian space-time.

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